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TWISTOR AND REFLECTOR SPACES OF ALMOST PARA-QUATERNIONIC MANIFOLDS

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Abstract. We investigate the integrability of natural almost complex structures on the twistor space of an almost para-quaternionic manifold as well as the integrability of natural almost paracomplex structures on the reflector space of an almost para-quaternionic manifold constructed with the help of a para-quaternionic connection. We show that if there is an integrable structure it is independent on the para-quaternionic connection. In dimension four, we express the anti-self-duality condition in terms of the Riemannian Ricci forms with respect to the associated para-quaternionic structure.

Key words: almost para-quaternionic mnifolds, anti-self-dual neutral metric, twistor space, almost complex structures.

MSC: 53C15, 5350, 53C25, 53C26, 53B30

Contents

1. Introduction and statement of the results

We study the geometry of structures on a differentiable manifold related to the algebra of paraquaternions. This structure leads to the notion of (almost) hyper-paracomplex and almost paraquaternionic manifolds in dimensions divisible by four. These structures are also attractive in theoretical physic since they play a role in string theory [32, 21, 7, 22, 14, 15] and integrable systems [16]. For example, hyper-paracomplex geometry arises in connection with different versions of the c-map [15]. New versions of the c-map are constructed in [15] which allow the authors to obtain the target manifolds of hypermultiplets in Euclidean theories with rigid N =2 supersymmetry. The authors show that the resulting hypermultiplet target spaces are para-hyper-Kähler manifolds.

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Both quaternions H and paraquaternions $\tilde{H}$ are real Clifford algebras, $H = C(2,0)$, $\tilde{H} = C(1,1) \cong C(0,2)$. In other words, the algebra $\tilde{H}$ of paraquaternions is generated by the unity 1 and the generators J_1^0, J_2^0, J_3^0 satisfying the *paraquaternionic identities*,

$$(J_1^0)^2 = (J_2^0)^2 = -(J_3^0)^2 = 1, \qquad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0. \tag{1.1}$$

We recall the notion of almost hyper-paracomplex manifold introduced by Libermann [31]. An *almost quaternionic structure of the second kind* on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 which mutually anti-commute, i.e. these structures satisfy the paraquaternionic identities (1.1). Such a structure is also called *complex product structure* [4, 3].

An *almost hyper-paracomplex structure* on a 4n-dimensional manifold M is a triple $\tilde{H} = (J_a), a = 1,2,3$, where J_α,$\alpha = 1,2$ are almost paracomplex structures $J_a : TM \to TM$, and $J_3 : TM \to TM$ is an almost complex structure, satisfying the paraquaternionic identities (1.1). We note that on an almost hyper-paracomplex manifold there is actually a 1-sheeted hyperboloid worth of almost complex structures:

$$S_1^2(-1) = \{c_1 J_1 + c_2 J_2 + c_3 J_3 : c_1^2 + c_2^2 - c_3^2 = -1\}$$

and a 2-sheeted hyperboloid worth of almost paracomplex structures:

$$S_1^2(1) = \{b_1 J_1 + b_2 J_2 + b_3 J_3 : b_1^2 + b_2^2 - b_3^2 = 1\}.$$

When each $J_a, a = 1,2,3$ is an integrable structure, $\tilde{H}$ is said to be a *hyper-paracomplex structure* on M. Such a structure is also called sometimes *pseudo-hyper-complex* [16].

It is well known that the structure J_a is integrable if and only if the corresponding Nijenhuis tensor $N_a = [J_a, J_a] + J_a^2[,] - J_a[J_a,] - J_a[,J_a]$ vanishes, $N_a = 0$. In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures $J_a, a = 1,2,3$ are integrable due to the existence of a linear identity between the three Nijenhuis tensors [26, 12]. In this case all almost complex structures of the two-sheeted hyperboloid $S_1^2(-1)$ as well as all paracomplex structures of the one-sheeted hyperboloid $S_1^2(1)$ are integrable. Examples of hyper-paracomplex structures on the simple Lie groups $SL(2n+1,\mathbb{R})$,$SU(n,n+1)$ are constructed in [24].

A *hyperparahermitian metric* is a pseudo Riemannian metric which is compatible with the (almost) hyperparacomplex structure $\tilde{H} = (J_a), a = 1,2,3$ in the sense that the metric is skew-symmetric with respect to each $J_a, a = 1,2,3$. Such a metric is necessarily of neutral signature (2n,2n). Such a structure is called *(almost) hyper-paraHermitian structure.*

An *almost para-quaternionic structure* on M is a rank-3 subbundle $\mathcal{P} \subset End(TM)$ which is locally spanned by an almost hyper-para-complex structure $\tilde{H} = (J_a)$; such a locally defined triple $\tilde{H}$ will be called admissible basis of $\mathcal{P}$. A linear connection ∇ on TM is called *para-quaternionic connection* if ∇ preserves $\mathcal{P}$. We denote the space all para-quaternionic connections on an almost para-quaternionic manifold by $\Delta(\mathcal{P})$.

An almost para-quaternionic structure is said to be a *para-quaternionic* if there is a torsion-free para-quaternionic connection.

An almost para-quaternionic (resp. para-quaternionic) manifold with hyperparahermitian metric is called an *almost para-quaternionic Hermitian* (resp. *para-quaternionic Hermitian*) manifold. If the Levi-Civita connection of a para-quaternionic Hermitian manifold is para-quaternionic connection, then the manifold is said to be *para-quaternionic Kähler* manifold.

This condition is equivalent to the statement that the holonomy group of g is contained in $Sp(n,\mathbf{R})Sp(1,\mathbf{R})$ for $n \geq 2$ [19, 35]. A typical example is the para-quaternionic projective space endowed with the standard para-quaternionic Kähler structure [11]. Any para-quaternionic Kähler manifold of dimension $4n \geq 8$ is known to be Einstein with scalar curvature s [19, 35]. If on a para-quaternionic Kähler manifold there exists an admissible basis $(\tilde{H})$ such that each $J_a, a = 1,2,3$ is parallel with respect to the Levi-Civita connection, then the manifold is said to be *hyper-paraKähler*. Such manifolds are also called *hypersymplectic* [20], *neutral hyper-Kähler* [28, 18]. The equivalent characterization is that the holonomy group of g is contained in $Sp(n,\mathbf{R})$ if $n \geq 2$ [35].

For $n = 1$ an almost para-quaternionic structure is the same as oriented neutral conformal structure [16, 19, 35, 12] and turns out to be always quaternionic. The existence of a (local) hyper-paracomplex structure is a strong condition since the integrability of the (local) almost hyper-paracomplex structure implies that the corresponding neutral conformal structure is anti-self-dual [1, 21, 26].

When $n \geq 2$, the para-quaternionic condition, i.e. the existence of torsion-free para-quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n,\tilde{H})\ Sp(1,\mathbf{R}) \cong GL(2n,\mathbf{R})\ Sp(1,\mathbf{R})$- structure [3, 4]. The paraquaternionic condition controls the Nijenhuis tensors in the sense that $NJ_a := N_a$ preserves the subbundle $\mathbb{P}$. An invariant first order differential operator D is defined on any almost paraquaternionic manifolds which is two-step nilpotent i.e. $D^2 = 0$ exactly when the structure is paraquaternionic [25]. Paraquaternionic structure is a type of a para-conformal structure [6] as well as a type of generalized hypercomplex structure [9].

Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold. The vector bundle $\mathcal{P}$ carries a natural Lorentz structure of signature (+,+,-) such that (J_1,J_2,J_3) forms an orthonormal local basis of P. There are two kinds of "unit sphere" bundles according to the existence of the 1-sheeted hyperboloid $S^2_1(1)$ and the 2-sheeted hyperboloid $S^2_1(-1)$. The *twistor space* $Z^-(M)$ is the unit pseudo-sphere bundle with fibre $S^2_1(1)$. The *reflector space* $Z^+(M)$ is the unit pseudo-sphere bundle with fibre $S^2_1(1)$. In other words, the fibre of $Z^-(M)$ consists of all almost complex structures compatible with the given paraquaternionic structure while the fibre of $Z^+(M)$ consists of all almost paracomplex structures compatible with the given paraquaternionic structure.

Keeping in mind the formal similarity with the quaternionic geometry where there are two natural almost complex structures on the corresponding twistor space [5, 17], one observes the existence of two naturally arising almost complex structures $I^{\nabla}_1, I^{\nabla}_2$ on $Z^-(M)$ and two almost paracomplex structures $P^{\nabla}_1, P^{\nabla}_2$ on $Z^+(M)$ defined with the help of the horizontal spaces of an arbitrary para-quaternionic connection $\nabla \in \Delta(\mathcal{P})$.

The almost paracomplex structures on the reflector space of a 4-dimensional manifold with neutral signature metric are defined using the horizontal spaces of the Levi-Civita connection in [27]. The authors show that one of the almost paracomplex structure is never integrable while the other almost paracomplex structure is integrable if and only if the neutral metric is anti-self-dual. The almost complex structures on the twistor space of a para-quaternionic Kähler manifold are defined and investigated in [10] using the horizontal spaces of the Levi-Civita connection. The authors show that one of the almost complex structure is never integrable while the other almost complex structure is always integrable. Both construction

are generalized in the case of twistor and reflector space of a para-quaternionic manifold in [26].

In the present note we investigate the dependence on the para-quaternionic connection of these structures on the twistor and reflector spaces over an almost para-quaternionic manifold. We obtain conditions on the paraquaternionic connection which imply the coincidence of the corresponding structures (Corollary 3.4, Corollary 3.3). We show that the existence of an integrable almost complex structure on the twistor space (resp. the existence of an integrable almost para-complex structure on the reflector space) does not depend on the para-quaternionic connection and it is equivalent to the condition that the almost para-quaternionic manifold is quaternionic provided the dimension is bigger than four (Theorem 3.8, Theorem 3.11).

In dimension four we find new relations between the Riemannian Ricci forms,i.e. the 2-forms which determine the $Sp(1,\mathbf{R})$-component of the Riemannian curvature, which are equivalent to the anti-self-duality of the oriented neutral conformal structure corresponding to a given para-quaternionic structure (Theorem 3.7).

In the last section we apply our considerations to the paraquaternionic Kähler manifold with torsion recently described by the third author in [36].

2. Preliminaries

Let $\tilde{\mathbf{H}}$ be the para-quaternions and identify $\tilde{\mathbf{H}}^n = \mathbf{R}^{4n}$. To fix notation we assume that $\tilde{\mathbf{H}}$ acts on $\tilde{\mathbf{H}}^n$ by right multiplication. This defines an antihomomorphism

$$\lambda : \{\mathrm{unit\ para-quaternions}\} = \\ = \{x + j_1 y + j_2 z + j_3 w \mid x^2 - y^2 - z^2 + w^2 = 1\} \longrightarrow SO(2n,2n) \subset GL(4n,\mathbf{R}),$$

where our convention is that $SO(2n,2n)$ acts on $\tilde{\mathbf{H}}^n$ on the left. Denote the image by $Sp(1,\mathbf{R})$ and let $J_1^0 = -\lambda(j_1), J_2^0 = -\lambda(j_2), J_3^0 = -\lambda(j_3)$. The Lie algebra of $Sp(1,\mathbf{R})$ is $sp(1,\mathbf{R}) = span\{J_1^0, J_2^0, J_3^0\}$ and we have

$$J_1^{0^2} = J_2^{0^2} = -J_3^{0^2} = 1, \qquad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0.$$

Define $GL(n,\tilde{H}) = \{A \in GL(4n,\mathbf{R}) : A(sp(1,\mathbf{R}))A^{-1} = sp(1,\mathbf{R})\}$. The Lie algebra of $GL(n,\tilde{H})$ is $gl(n,\tilde{H}) = \{A \in gl(4n,\mathbf{R}) : AB = BA$ for all $B \in sp(1,\mathbf{R})\}$.

Let $(M,\mathcal{P})$ be an almost paraquaternionic manifold and $\tilde{H} = (J_a), a = 1,2,3$ be an admissible local basis. We shall use the notation $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$.

Let $B \in \Lambda^2(TM)$. We say that B is of type $(0,2)_{J_a}$ with respect to J_a if

$$B(J_a X, Y) = -J_a B(X,Y)$$

and denote this space by $\Lambda^{0,2}_{J_a}$. The projection $B^{0,2}_{J_a}$ is given by

$$B^{0,2}_{J_a}(X,Y) = \frac{1}{4}\left(\epsilon_a B(X,Y) + B(J_a X, J_a Y) - J_a B(J_a X, Y) - J_a B(X, J_a Y)\right).$$

For example, the Nijenhuis tensor $N_a \in \Lambda^{0,2}_{J_a}$.

Let $\nabla \in \Delta(\mathcal{P})$ be a para-quaternionic connection on an almost paraquaternionic manifold $(M, \mathcal{P})$. This means that there exist locally defined 1-forms $\omega_a, a = 1, 2, 3$ such that zzv

$$\begin{aligned}
\nabla J_1 &= -\omega_3 \otimes J_2 + \omega_2 \otimes J_3, \\
\nabla J_2 &= \omega_3 \otimes J_1 + \omega_1 \otimes J_3, \\
\nabla J_3 &= \omega_2 \otimes J_1 + \omega_1 \otimes J_2.
\end{aligned} \tag{2.2}$$

An easy consequence of (2.2) is that the curvature R^∇ of any para-quaternionic connection $\nabla \in \Delta(\mathcal{P})$ satisfies the relations

$$\begin{aligned}
[R^\nabla, J_1] &= -A_3 \otimes J_2 + A_2 \otimes J_3, \qquad A_1 = d\omega_1 + \omega_2 \wedge \omega_3 \\
[R^\nabla, J_2] &= A_3 \otimes J_1 + A_1 \otimes J_3, \qquad A_2 = d\omega_2 + \omega_3 \wedge \omega_1, \\
[R^\nabla, J_3] &= A_2 \otimes J_1 + A \otimes J_2, \qquad A_3 = d\omega_3 - \omega_1 \wedge \omega_2.
\end{aligned} \tag{2.3}$$

The Ricci 2-forms of a para-quaternionic connection are defined by

$$\begin{aligned}
\rho^\nabla_\alpha(X, Y) &= \frac{1}{2} Tr(Z \longrightarrow J_a R^\nabla(X, Y)Z), \quad \alpha = 1, 2, \\
\rho^\nabla_3(X, Y) &= -\frac{1}{2} Tr(Z \longrightarrow J_3 R^\nabla(X, Y)Z).
\end{aligned}$$

It is easy to see using (2.3) that the Ricci forms are given by

$$\rho^\nabla_1 = d\omega_1 + \omega_2 \wedge \omega_3, \quad \rho^\nabla_2 = -d\omega_2 - \omega_3 \wedge \omega_1, \quad \rho^\nabla_3 = d\omega_3 - \omega_1 \wedge \omega_2.$$

We split the curvature of ∇ into $gl(n, \tilde{H})$-valued part $(R^\nabla)'$ and $sp(1, \mathbf{R})$-valued part $(R^\nabla)''$ following the classical scheme (see e.g. [2, 23, 8])

Proposition 2.1. *The curvature of an almost para-quaternionic connection on M splits as follows*

$$\begin{aligned}
R^\nabla(X, Y) &= (R^\nabla)'(X, Y) + \frac{1}{2n}(\rho^\nabla_1(X, Y)J_1 + \rho^\nabla_2(X, Y)J_2 + \rho^\nabla_3(X, Y)J_3), \\
& [(R^\nabla)'(X, Y), J_a] = 0, \quad a = 1, 2, 3,
\end{aligned}$$

Let Ω, Θ be the curvature 2-form and the torsion 2-form of ∇ on $P(M)$, respectively (see e.g. [29]). We denote the splitting of the $gl(n, \tilde{H}) \oplus sp(1, \mathbf{R})$-valued curvature 2-form Ω on $P(M)$ according to Proposition 2.1, by $\Omega = \Omega' + \Omega''$, where Ω' is a $gl(n, \tilde{H})$-valued 2-form and Ω'' is a $sp(1, \mathbf{R})$-valued form. Explicitly,

$$\Omega'' = \Omega''_1 J^0_1 + \Omega''_2 J^0_2 + \Omega''_3 J^0_3,$$

where $\Omega''_a, a = 1, 2, 3$, are 2-forms. If $\xi, \eta, \zeta \in \mathbf{R^{4n}}$, then the 2-forms $\Omega''_a, a = 1, 2, 3$, are given by

$$\Omega''_a(B(\xi), B(\eta)) = \frac{1}{2n} \rho_a(X, Y), \quad X = u(\xi), Y = u(\eta).$$

3. Twistor and reflector spaces of almost para-quaternionic manifolds

Consider the space $\tilde{H}_1$ of imaginary para-quaternions. It is isomorphic to the Lorentz space $\mathbf{R^2_1}$ with a Lorentz metric of signature (+,+,-) defined by $< q, q' >= -Re(q\overline{q'})$, where $\overline{q} = -q$ is the conjugate imaginary para-quaternion. In $\mathbf{R^2_1}$ there are two kinds of 'unit spheres', namely the pseudo-sphere $S^2_1(1)$ of radius 1 (the 1-sheeted hyperboloid) which consists of all imaginary para-quaternions of norm 1 and the pseudo-sphere $S^2_1(-1)$ of radius (-1) (the 2-sheeted hyperboloid) which contains all imaginary para-quaternions of norm (-1). The 1-sheeted hyperboloid $S^2_1(1)$ carries a natural para-complex structure while the 2-sheeted hyperboloid $S^2_1(-1)$ carries a natural complex structure, both induced by the cross-product on $\tilde{H}_1 \cong \mathbf{R^2_1}$ defined by

$$X \times Y = \sum_{i \neq k} x^i y^k J_i J_k$$

for vectors $X = x^i J_i, \quad Y = y^k J_k$. Namely, for a tangent vector $X = x^i J_i$ to the 1-sheeted hyperboloid $S^2_1(1)$ at a point $q_+ = q^k_+ J_k$ (resp. tangent vector $Y = y^k_- J_k$ to the 2-sheeted hyperboloid $S^2_1(-1)$ at a point $q_- = q^k_- J_k$) we define $PX := q_+ \times X$ (resp. $JY = q_- \times Y$). It is easy to check that PX is again tangent vector to $S^2_1(1) \quad and \quad P^2 X = X$ (resp. JY is tangent vector to $S^2_1(-1) \quad and \quad J^2 Y = -Y$).

Let M be a $4n$-dimensional manifold endowed with an almost para-quaternionic structure $\mathcal{P}$. Let J_1, J_2, J_3 be an admissible basis of $\mathcal{P}$ defined in some neighborhood of a given point $p \in M$. Any linear frame u of T_pM can be considered as an isomorphism $u : \mathbf{R}^{4n} \longrightarrow T_pM$. If we pick such a frame u we can define a subspace of the space of the all endomorphisms of T_pM by $u(sp(1, \mathbf{R}))u^{-1}$. Clearly, this subset is a para-quaternionic structure at the point p and in the general case this para-quaternionic structure is different from $\mathcal{P}_p$. We define $P(M)$ to be the set of all linear frames u which satisfy $u(sp(1, \mathbf{R}))u^{-1} = \mathcal{P}$. It is easy to see that $P(M)$ is a principal frame bundle of M with structure group $GL(n, \tilde{H})Sp(1, \mathbf{R})$, it is also called a $GL(n, \tilde{H})Sp(1, \mathbf{R})$-structure on M.

Let $\pi : P(M) \longrightarrow M$ be the natural projection. For each $u \in P(M)$ we consider two linear isomorphisms $j^+(u)$ and $j^-(u)$ on $T_{\pi(u)}M$ defined by $j^+(u) = uJ^0_1 u^{-1}$ and $j^-(u) = uJ^0_3 u^{-1}$. It is easy to see that $(j(u)^+)^2 = id$ and $(j(u)^-)^2 = -id$. For each point $p \in M$ we define $Z^+_p(M) = \{j^+(u) : u \in P(M), \pi(u) = p\}$ and $Z^-_p(M) = \{j^-(u) : u \in P(M), \pi(u) = p\}$. In other words, $Z^-_p(M)$ is the connected component of J_3 of the space of all complex structures (resp. $Z^+_p(M)$ is the space of all para-complex structures) in the tangent space T_pM which are compatible with the almost para-quaternionic structure on M.

We define the twistor space Z^- of M, by setting $Z^- = \bigcup_{p \in M} Z^-_p(M)$. Let H_3 be the stabilizer of J^0_3 in the group $GL(n, \tilde{H})Sp(1, \mathbf{R})$. There is a bijective correspondence between the symmetric space $GL(n, \tilde{H})Sp(1, \mathbf{R})/H_3 \cong S^2_1(-1)^+ = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 - z^2 = -1, \ z > 0\}$ and $Z^-_p(M)$ for each $p \in M$. So we can consider Z^- as the associated fibre bundle of $P(M)$ with standard fibre $GL(n, \tilde{H})Sp(1, \mathbf{R})/H_3$. Hence, $P(M)$ is a principal fibre bundle over Z^- with structure group H_3 and projection j^-. We consider the symmetric spaces $GL(n, \tilde{H})Sp(1, \mathbf{R})/H_3$. We have the following Cartan decomposition $gl(n, \tilde{H}) \oplus sp(1, \mathbf{R}) = h_3 \oplus m_3$ where

$$h_3 = \{A \in gl(n, \tilde{H}) \oplus sp(1, \mathbf{R}) : AJ^0_3 = J^0_3 A\}$$

is the Lie algebra of H_3 and $m_3 = \{A \in gl(n,\tilde{H}) \oplus sp(1,\mathbf{R}) : AJ_3^0 = -J_3^0 A\}$. It is clear that m_3 is generated by J_1^0, J_2^0, i.e. $m_3 = span\{J_1^0, J_2^0\}$. Hence, if $A \in m_3$ then $J_3^0 A \in m_3$.

We proceed with defining the reflector space Z^+ of M. We put $Z^+ = \bigcup_{p\in M} Z_p^+(M)$. Let H_1 be the stabilizer of J_1^0 in the group $GL(n,\tilde{H})Sp(1,\mathbf{R})$. There is a bijective correspondence between the symmetric space $GL(n,\tilde{H})\ Sp(1,\mathbf{R})/H_1 \cong S_1^2(1) = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 - z^2 = 1\}$ and $Z_p^+(M)$ for each $p \in M$. So we can consider Z^+ as the associated fibre bundle of $P(M)$ with standard fibre $GL(n,\tilde{H})Sp(1,\mathbf{R})/H_1$. Hence, $P(M)$ is a principal fibre bundle over Z^+ with structure group H_1 and projection j^+. We consider the symmetric spaces $GL(n,\tilde{H})Sp(1,\mathbf{R})/H_1$. We have the following Cartan decomposition $gl(n,\tilde{H}) \oplus sp(1,\mathbf{R}) = h_1 \oplus m_1$ where

$$h_1 = \{A \in gl(n,\tilde{H}) \oplus sp(1,\mathbf{R}) : AJ_1^0 = J_1^0 A\}$$

is the Lie algebra of H_1 and $m_1 = \{A \in gl(n,\tilde{H}) \oplus sp(1,\mathbf{R}) : AJ_1^0 = -J_1^0 A\}$. It is clear that m_1 is generated by J_2^0, J_3^0, i.e. $m_1 = span\{J_2^0, J_3^0\}$. Hence, if $A \in m_1$ then $J_1^0 A \in m_1$.

Let ∇ be a para-quaternionic connection on M, i.e. ∇ is a linear connection in the principal bundle $P(M)$ according to ([29]). Note that we make no assumptions on the torsion or on the curvature of ∇. Keeping in mind the formal similarity with the quaternionic geometry where one uses a quaternionic connection to define two natural almost complex structures on the corresponding twistor space [5, 17, 33, 34], we use ∇ to define two almost complex structures I_1^∇ and I_2^∇ on the twistor space Z^- and two almost para-complex structures P_1^∇ and P_2^∇ on the reflector space Z^+. Apparently, the construction of these structures depends on the choice of the para-quaternionic connection ∇.

We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on $P(M)$ corresponding to $A \in gl(n,\tilde{H}) \oplus sp(1,\mathbf{R})$ (resp. $\xi \in \mathbf{R^{4n}}$).

Let $u \in P(M)$ and Q_u be the horizontal subspace of the tangent space $T_u P(M)$ induced by ∇ (see e.g. [29]). The vertical space i.e. the vector space tangent to a fibre is isomorhic to

$$(gl(n,\tilde{H}) \oplus sp(1,\mathbf{R}))_u^* = (h_3)_u^* \oplus (m_3)_u^* = (h_1)_u^* \oplus (m_1)_u^*,$$

where $(h_i)_u^* = \{A_u^* : A \in h_i\}, (m_i)_u^* = \{A_u^* : A \in m_i\},\ i = 1,3.$

Hence, $T_u P(M) = (h_i)_u^* \oplus (m_i)_u^* \oplus Q_u$.

For each $u \in P(M)$ we put

$$V^-_{j^-(u)} = j^-_{*u}((m_3)_u^*), H^-_{j^-(u)} = j^-_{*u} Q_u \quad V^+_{j^+(u)} = j^+_{*u}((m_1)_u^*), H^+_{j(u)} = j^+_{*u} Q_u.$$

Thus we obtain vertical and horizontal distributions V^- and H^- on Z^-(resp. V^+ and H^+ on Z^+). Since $P(M)$ is a principal fibre bundle over Z^- (resp. Z^+) with structure group H_3 (resp H_1) we have $Kerj^-_{*u} = (h_3)_u^*$ (resp. $Kerj^+_{*u} = (h_1)_u^*$).

Hence $V^-_{j^-(u)} = j^-_{*u}(m_3)_u^*$ and $j^-_{*u|(m_3)_u^* \oplus Q_u} : (m_3)_u^* \oplus Q_u \longrightarrow T_{j^-(u)} Z^-$ is an isomorphism (resp. $V^+_{j^+(u)} = j^+_{*u}(m_1)_u^*$ and $j^+_{*u|(m_1)_u^* \oplus Q_u} : (m_1)_u^* \oplus Q_u \longrightarrow T_{j^+(u)} Z^+$ is an isomorphism).

We define two almost complex structures I_1^∇ and I_2^∇ on Z^- by

$$\begin{aligned} I_1^\nabla(j^-_{*u} A^*) &= j^-_{*u}(J_3^0 A)^*, \qquad I_2^\nabla(j^-_{*u} A^*) = -j^-_{*u}(J_3^0 A)^* \\ I_i^\nabla(j^-_{*u} B(\xi)) &= j^-_{*u} B(J_3^0 \xi), \qquad i = 1,2, \end{aligned} \tag{3.4}$$

for $A \in m_3, \xi \in \mathbf{R^{4n}}$.

Similarly, we define two almost para-complex structures P_1^{∇} and P_2^{∇} on Z^+ by

$$
\begin{aligned}
&P_1^{\nabla}(j^+_{*u}A^*) = j^+_{*u}(J_1^0A)^*, \qquad P_2^{\nabla}(j^+_{*u}A^*) = -j^+_{*u}(J_1^0A)^* \\
&P_i^{\nabla}(j^+_{*u}B(\xi)) = j^+_{*u}B(J_1^0\xi), \qquad i=1,2,
\end{aligned}
\tag{3.5}
$$

for $A \in m_1, \xi \in \mathbf{R^{4n}}$.

The almost paracomplex structures (3.5) on the reflector space of a 4-dimensional manifold with neutral signature metric are defined using the horizontal spaces of the Levi-Civita connection ∇^g in [27]. The authors show that the almost paracomplex structure $P_2^{\nabla^g}$ is never integrable while the almost paracomplex structure $P_1^{\nabla^g}$ is integrable if and only if the neutral metric is anti-self-dual. The almost complex structures (3.4) on the twistor space of a para-quaternionic Kähler manifold are defined and investigated in [10] with the help of the horizontal spaces of the Levi-Civita connection. The authors show that the almost complex structure $I_2^{\nabla^g}$ is never integrable while the almost complex structure $I_1^{\nabla^g}$ is always integrable. Both construction are generalized in the case of twistor and reflector space of a para-quaternionic manifold in [26]. Twistor space of para-quaternionic Kähler manifold is investigated also in [13] where the LeBrun's inverse twistor construction for quaternionic Kähler manifolds [30] has been adapted to the case of para-quaternionic Kähler manifolds.

We finish this section with the next useful

Lemma 3.1. *Let $J_- \in Z^-$ be an almost complex structure or $J_+ \in Z^+$ be an almost para-complex structure and $B \in \Lambda^2(TM)$.*

If $B^{0,2}_{J_-} = 0$ for all $J_- \in Z_-$ then $B^{0,2}_{J_+} = 0$ for all $J_+ \in Z_+$ and vice versa.

Proof. Let $J_t = \sinh tJ_1 + \cosh tJ_3, t \in \mathbb{R}$ be an almost complex structure in Z_-. Using the conditions $B^{0,2}_{J_t} = 0 = B^{0,2}_{J_3}$, we calculate

$$\frac{1}{2}(1+\cosh 2t)B^{0,2}_{J_1} + \frac{1}{2}\sinh 2t[\mathcal{B}] = 0,$$

where $\mathcal{B}$ is a tensor field depending on B.

The latter leads to $B^{0,2}_{J_1} = 0$. Similarly, $B^{0,2}_{J_2} = 0$ and the lemma follows. $\square$

3.1. **Dependence on the para-quaternionic connection.** In this section we investigate when different almost para-quaternionic connections induce the same structure on the twistor or reflector space over an almost para-quaternionic manofold.

Let ∇ and ∇' be two different almost para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. Then we have

$$\nabla'_X = \nabla_X + S_X, \qquad X \in \Gamma(TM),$$

where S_X is a (1,1) tensor on M and $u^{-1}(S_X)u$ belongs to $gl(n, \tilde{H}) \oplus sp(1, \mathbf{R})$ for any $u \in P(M)$. Thus we have the splitting

$$S_X(Y) = S^0_X(Y) + s^1(X)J_1Y + s^2(X)J_2Y + s^3(X)J_3Y, \tag{3.6}$$

where $X, Y \in \Gamma(TM)$, s^i are 1-forms and $[S^0_X, J_i] = 0$, $i = 1,2,3$.

Proposition 3.2. *Let ∇ and ∇' be two different para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. The following conditions are equivalent:*

i). The two almost complex structures I_1^{∇} and $I_1^{\nabla'}$ on the twistor space Z^- coincide.

ii). The 1-forms s^1, s^2, s^3 are related as follows

$$s^1(J_1X) = s^2(J_2X) = s^3(J_3X), \qquad X \in \Gamma(TM).$$

iii). The two almost para-complex structures P_1^{∇} and $P_1^{\nabla'}$ on the reflector space Z^+ coincide.

Proof. We fix a point J of the twistor space Z^-. We have $J = a_1J_1 + a_2J_2 + a_3J_3$ with $a_1^2 + a_2^2 - a_3^2 = -1$. Let $\pi : Z^- \longrightarrow M$ be the natural projection and $x = \pi(J)$. The connection ∇ induces a splitting of the tangent space of Z^- into vertical and horizontal components: $T_JZ^- = V_J^- \oplus H_J^-$. Let v and h be the vertical and horizontal projections corresponding to this splitting. Let $T_JZ^- = {V'}_J^- \oplus {H'}_J^-$ be the splitting induced by ∇' with the projections v' and h', respectively. It is easy to observe the following identities

$$\begin{aligned} &v + h = 1 \\ &v' + h' = 1 \\ &vv' = v' \\ &v' + vh' = v \end{aligned} \tag{3.7}$$

In fact, $V_J^- = {V'}_J^-$ and we may regard this space as a subspace of $\mathcal{P}_x$. We have that

$$V_J^- = \{W \in \mathcal{P}_x \mid WJ + JW = 0\} = \{w_1J_1 + w_2J_2 + w_3J_3 \mid w_1a_1 + w_2a_2 - w_3a_3 = 0\},$$

where $J = a_1J_1 + a_2J_2 + a_3J_3$. It follows that for any $W \in V_J^-$, $I_1^{\nabla}(W) = I_1^{\nabla'}(W) = JW$. In general, for any $W \in T_JZ^-$, we have

$$\begin{aligned} I_1^{\nabla}(W) &= J(vW) + (J\pi(W))^h \\ I_1^{\nabla'}(W) &= J(v'W) + (J\pi(W))^{h'}, \end{aligned}$$

where $(.)^h$ (resp. $(.)^{h'}$) denotes the horizontal lift on Z^- of the corresponding vector field on M with respect to ∇ (resp. ∇'). Using (3.7), we calculate that

$$\begin{aligned} v(I_1^{\nabla'}W) = J(v'W) + v(J\pi(W))^{h'} = J((v - vh')W) + v(J\pi(W))^{h'} = \\ v(I_1^{\nabla}W) - J(vh'W) + v(J\pi(W))^{h'}. \end{aligned} \tag{3.8}$$

We investigate the equality

$$J(vh'W) = v(J\pi(W))^{h'}, \qquad W \in T_JZ^-. \tag{3.9}$$

Take $W = Y^{h'}, Y \in \Gamma(TM)$ in (3.9) to get

$$J(vY^{h'}) = v(JY)^{h'}, \qquad Y \in T_xM \tag{3.10}$$

Hence, (3.10) is equivalent to $I_1^{\nabla} = I_1^{\nabla'}$ because of (3.8).

Let $(U, x_1, \ldots, x_{4n})$ be a local coordinate system on M and let $Y = \sum Y^i \frac{\partial}{\partial x^i}$. The horizontal lift of Y with respect to ∇' at the point $J \in Z^-$ is given by

$$Y_J^{h'} = \sum_{i=1}^{4n} (Y^i \circ \pi) \frac{\partial}{\partial x^i} - \sum_{s=1}^{3} a_s {\nabla'}_Y J_s$$

We calculate

$$v(JY)^{h'} = (JY)^{h'} - h(JY)^{h'} = (JY)^{h'} - (JY)^h = \tag{3.11}$$
$$= \sum_{s=1}^{3} a_s(-\nabla'_{JY}J_s + \nabla_{JY}J_s) = -[S_{JY}, J]$$

On the other hand, we have

$$J(vY^{h'}) = J(Y^{h'} - Y^h) = J\sum_{s=1}^{3} a_s(-\nabla'_{Y}J_s + \nabla_{Y}J_s) = -J[S_Y, J] \tag{3.12}$$

Substitute (3.11) and (3.12) into (3.10) to get that $I_1^{\nabla} = I_1^{\nabla'}$ is equivalent to the condition

$$J[S_Y, J] = [S_{JY}, J], \qquad Y \in \Gamma(TM), J \in Z^-. \tag{3.13}$$

Now, (3.13) and (3.6) easily lead to the equivalence of i) and ii).

Similarly, we obtain that $P_1^{\nabla} = P_1^{\nabla'}$ if and only if

$$P[S_Y, J] = [S_{PY}, J] \tag{3.14}$$

for any choice of $P \in Z^+$ and $Y \in TM$.

The equality (3.14) together with (3.6) implies the equivalence of ii) and iii). □

Corollary 3.3. *Let ∇ and ∇' be two different para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. The following conditions are equivalent:*

i). The two almost complex structures I_2^{∇} and $I_2^{\nabla'}$ on the twistor space Z^- coincide.
ii). The 1-forms s^1, s^2, s^3 vanish, $s_1 = s_2 = s_3 = 0$.
iii). The two almost para-complex structures P_2^{∇} and $P_2^{\nabla'}$ on the reflector space Z^+ coincide.

Proof. It is sufficient to observe from the proof of Proposition 3.2 that $I_2^{\nabla} = I_2^{\nabla'}$ is equivalent to $J[S_Y, J] = -[S_{JY}, J], \quad Y \in \Gamma(TM), J \in Z^-$ while $P_1^{\nabla} = P_1^{\nabla'}$ if and only if $P[S_Y, J] = -[S_{PY}, J]$ for any choice of $P \in Z^+$ and $Y \in TM$. Each one of the last two conditions imply $s_1 = s_2 = s_3 = 0$. □

Corollary 3.4. *Let ∇ and ∇' be two different para-quaternionic connections with torsion tensors $T^{\nabla'}$ and T^{∇}, respectively, on an almost para-quaternionic manifold $(M, \mathcal{P})$. The following conditions are equivalent:*

i). The two almost complex structures I_1^{∇} and $I_1^{\nabla'}$ on the twistor space Z^- coincide.
ii). The $(0,2)_J$ part with respect to all $J \in \mathcal{P}$ of the torsion T^{∇} and $T^{\nabla'}$ coincides, $(T^{\nabla})_J^{0,2} = (T^{\nabla'})_J^{0,2}$.
iii). The two almost para-complex structures P_1^{∇} and $P_1^{\nabla'}$ on the reflector space Z^+ coincide.

Proof. The equivalence of i) and iii) has been proved in Proposition 3.4.

Let $S = \nabla' - \nabla$. Then we have

$$T^{\nabla'}(X, Y) = T^{\nabla}(X, Y) + S_X(Y) - S_Y(X). \tag{3.15}$$

The $(0,2)_J$-part with respect to J of (3.15) gives

$$(T^{\nabla'})^{0,2}_J - (T^{\nabla})^{0,2}_J = [S_{JX}, J]Y - J[S_X, J]Y - [S_{JY}, J]X + J[S_Y, J]X. \tag{3.16}$$

Suppose iii) holds. Substitute (3.14) into the right hand side of (3.16) and use Lemma 3.1 to get $(T^{\nabla'})^{0,2}_J = (T^{\nabla})^{0,2}_J$, i.e. ii) is true.

For the converse, put $J = J_2$ in (3.16) and use the splitting (3.6) to obtain

$$\begin{aligned} \frac{1}{2}\left(T^{\nabla'})^{0,2}_{J_2} - (T^{\nabla})^{0,2}_{J_2}\right) &= [s_1(X) + s_3(J_2X)]\, J_1Y + [s_1(J_2X) + s_3(X)]\, J_3Y \\ &- [s_1(Y) + s_3(J_2Y)]\, J_1X - [s_1(J_2Y) + s_3(Y)]\, J_3X. \end{aligned} \tag{3.17}$$

Hence, $s_1(J_1X) = s_3(J_3X)$ is equivalent to $(T^{\nabla'})^{0,2}_{J_2} = (T^{\nabla})^{0,2}_{J_2}$ Substitute $J = J_1$ in (3.16) and use the splitting (3.6) to obtain $s_2(J_2X) = s_3(J_3X)$ is equivalent to $(T^{\nabla'})^{0,2}_{J_1} = (T^{\nabla})^{0,2}_{J_1}$. Now, Lemma 3.1 together with Proposition 3.2 completes the proof. □

3.2. **Integrability.**

In this section we investigate conditions on the para- quaternionic connection ∇ which imply the integrability of the almost complex structure I_1^{∇} on Z^- and almost para-complex structure P_1^{∇} on Z^+. We also show that I_2^{∇} and P_2^{∇} are never integrable i.e. for any choice of the para-quaternionic connection ∇ each of these two structures has non-vanishing Nijenhuis tensor.

We denote by $IN_i, PN_i, \quad i = 1,2$ the Nijenhuis tensors of I_i and P_i, respectively and recall that

$$\begin{aligned} IN_i(U,W) &= [I_iU, I_iW] - [U,W] - I_i[I_iU,W] - I_i[U,I_iW], \quad U,W \in \Gamma(TZ^-), \\ PN_i(U,W) &= [P_iU, P_iW] + [U,W] - P_i[P_iU,W] - P_i[U,P_iW], \quad U,W \in \Gamma(TZ^+). \end{aligned}$$

Proposition 3.5. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic manifold $(M, \mathcal{P})$ with torsion tensor T^{∇}. The following conditions are equivalent:*

i). The almost complex structure I_1^{∇} on the twistor space Z^- of $(M,\mathcal{P})$ is integrable.

ii). The $(0,2)_J$-part $(T^{\nabla})^{0,2}_J$ of the torsion with respect to all $J \in \mathcal{P}$ vanishes,

$$(T^{\nabla})^{0,2}_J = 0, J \in \mathcal{P} \tag{3.18}$$

and the (2,0)+(0,2) parts of the Ricci 2-forms with respect to an admissible basis J_1, J_2, J_3 of $\mathcal{P}$ coincide in the sense that the following identities hold

$$\rho_a(J_bX, J_bY) + \epsilon_b\rho_a(X,Y) - \epsilon_c\rho_c(J_bX,Y) - \epsilon_c\rho_c(X,J_bY) = 0, \tag{3.19}$$

where $\{a,b,c\}$ is a cyclic permutation of $\{1,2,3\}$ and $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$.

iii). The almost paracomplex structure P_1^{∇} on the reflector space Z^+ of $(M,\mathcal{P})$ is integrable.

Proof. Let J_1, J_2, J_3 be an admissible basis of the almost para-quaternionic structure $\mathcal{P}$.

Let *hor* be the natural projection $T_uP \longrightarrow (m_3)^*_u \oplus Q_u$, with $ker(hor) = (h_3)^*_u$. We define a tensor field I_1' on $P(M)$ by

$$\begin{aligned} &I_1'(U) \in (m_3)^*_u \oplus Q_u, \\ &(j^-)_{*u}(I_1'(U)) = I_1((j^-)_{*u}U), \qquad U \in T_uP. \end{aligned}$$

For any $U, W \in \Gamma(TP(M))$ we define

$$IN_1^{'}(U,W) = hor[I_1^{'}U, I_1^{'}W] - hor[horU, horW] - I_1^{'}[I_1^{'}U, horW] - I_1^{'}[horU, I_1^{'}W]$$

It is easy to check that $IN_1^{'}$ is a a tensor field on $P(M)$. We also observe that

$$j^-_{*u}(IN_1^{'}(U,W)) = IN_1(j^-_{*u}U, j^-_{*u}W), \qquad U, W \in T_uP(M) \tag{3.20}$$

Let $A, B \in m_3$ and $\xi, \eta \in \mathbf{R}^{4n}$. Using the well known general commutation relations among the fundamental vector fields and standard horizontal vector fields on the principal bundle $P(M)$ (see e.g. [29]), we calculate taking into account (3.20) that

$$IN_1(j^-_{*u}(A^*_u), j^-_{*u}(B^*_u)) = 0.$$
$$IN_1(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = 0.$$
$$\begin{aligned}[IN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{H^-} = \\ j^-_{*u}(B(-\Theta(B(J_3^0\xi), B(J_3^0\eta)) + \Theta(B(\xi), B(\eta)) \\ +J_3^0\Theta(B(J_3^0\xi), B(\eta)) + J_3^0\Theta(B(\xi), B(J_3^0\eta)))_u).\end{aligned} \tag{3.21}$$
$$\begin{aligned}[IN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{V^-} = \\ \{-\rho_1(B(J_3^0\xi), B(J_3^0\eta)) + \rho_1(B(\xi), B(\eta)) \\ +\rho_2(B(J_3^0\xi), B(\eta)) + \rho_2(B(\xi), B(J_3^0\eta))\}j^-_{*u}(J_1^0) \\ +\{-\rho_2(B(J_3^0\xi), B(J_3^0\eta)) + \rho_2(B(\xi), B(\eta)) \\ -\rho_1(B(J_3^0\xi), B(\eta)) - \rho_1(B(\xi), B(J_3^0\eta)))\}j^-_{*u}(J_2^0).\end{aligned} \tag{3.22}$$
$$IN_2(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = -4j^-_{*u}(B(A\xi)_u) \neq 0. \tag{3.23}$$

Concerning the reflector space, let $horr$ be the natural projection $T_uP \longrightarrow (m_1)^*_u \oplus Q_u$, with $ker(horr) = (h_1)^*_u$. In a very similar way as above, we calculate

$$PN_1(j^-_{*u}(A^*_u), j^-_{*u}(B^*_u)) = 0.$$
$$PN_1(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = 0$$
$$\begin{aligned}[PN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{H^-} = \\ j^-_{*u}(B(-\Theta(B(J_1^0\xi), B(J_1^0\eta)) - \Theta(B(\xi), B(\eta)) \\ +J_1^0\Theta(B(J_1^0\xi), B(\eta)) + J_1^0\Theta(B(\xi), B(J_1^0\eta)))_u).\end{aligned} \tag{3.24}$$
$$\begin{aligned}[PN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{V^-} = \\ \{-\rho_2(B(J_1^0\xi), B(J_1^0\eta)) - \rho_2(B(\xi), B(\eta)) \\ +\rho_3(B(J_1^0\xi), B(\eta)) + \rho_3(B(\xi), B(J_1^0\eta))\}j^-_{*u}(J_2^0) \\ +\{-\rho_3(B(J_1^0\xi), B(J_1^0\eta)) - \rho_3(B(\xi), B(\eta)) \\ +\rho_2(B(J_1^0\xi), B(\eta)) + \rho_2(B(\xi), B(J_1^0\eta))\}j^-_{*u}(J_3^0).\end{aligned} \tag{3.25}$$
$$PN_2(j^+_{*u}(A^*_u), j^+_{*u}(B(\xi)_u)) = 4j^+_{*u}(B(A\xi)_u) \neq 0. \tag{3.26}$$

Take $X = u(\xi), Y = u(\eta)$ we see that (3.21), (3.22), (3.24) and (3.25) are equivalent to

$$(T^{\nabla})^{0,2}_{J_3} = T^{\nabla}(J_3X, J_3Y) - T^{\nabla}(X, Y) - J_3T^{\nabla}(J_3X, Y) - J_3T^{\nabla}(X, J_3Y) = 0, \tag{3.27}$$

$$\rho^{\nabla}_1(J_3X, J_3Y) - \rho^{\nabla}_1(X, Y) - \rho^{\nabla}_2(J_3X, Y) - \rho^{\nabla}_2(X, J_3Y) = 0, \tag{3.28}$$

$$(T^{\nabla})^{0,2}_{J_1} = T^{\nabla}(J_1X, J_1Y) + T^{\nabla}(X, Y) - J_1T^{\nabla}(J_1X, Y) - J_1T^{\nabla}(X, J_1Y) = 0, \tag{3.29}$$

$$\rho^{\nabla}_3(J_1X, J_1Y) + \rho^{\nabla}_3(X, Y) - \rho^{\nabla}_2(J_1X, Y) - \rho^{\nabla}_2(X, J_1Y) = 0, \tag{3.30}$$

respectively.

With the help of Lemma 3.1, we see that (3.27) as well as (3.29) is equivalent to the statement $(T^{\nabla})^{0,2}_J = 0$ for all local $J \in \mathcal{P}$. To complete the proof we observe that each of the equalities (3.28) and (3.30) is equivalent to (3.19). □

The equations (3.23) and (3.26) in the proof of Proposition 3.5 yield

Corollary 3.6. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic manifold $(M, \mathcal{P})$ with torsion tensor T^{∇}.*

(1) The almost complex structure I^{∇}_2 on the twistor space Z^- of $(M, \mathcal{P})$ is never integrable.
(2) The almost paracomplex structure P^{∇}_2 on the reflector space Z^+ of $(M, \mathcal{P})$ is never integrable.

In the 4-dimensional case we derive

Theorem 3.7. *Let (M^4, g) be a 4-dimensional pseudo-Riemannian manifold with neutral metric g and let $\mathcal{P}$ be the para-quaternionic structure corresponding to the conformal class generated by g with a local basis J_1, J_2, J_3. Then the following conditions are equivalent*

i). The neutral metric g is anti-self-dual.
ii). The Ricci forms ρ^g_a of the Levi-Civita connection ∇^g satisfy (3.19), i.e.

$$\rho^g_a(J_bX, J_bY) + \epsilon_b\rho^g_a(X, Y) - \epsilon_c\rho^g_c(J_bX, Y) - \epsilon_c\rho^g_c(X, J_bY) = 0.$$

iii). The torsion condition (3.18) for a linear connection ∇ always implies the curvature condition (3.19).

Proof. The proof is a direct consequence of Proposition 3.5, Corolarry 3.4 and the result in [27] (resp. [10]) which states that the almost para-complex structure $P^{\nabla^g}_1$ (resp. the almost complex structure $I^{\nabla^g}_1$) is integrable exactly when the neutral conformal structure generetaed by g is anti-self-dual. □

In higher dimensions, the curvature condition (3.19) is a consequence of the torsion condition (3.18) in the sense of the next

Theorem 3.8. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic $4n$-dimensional $n \geq 2$ manifold $(M, \mathcal{P})$ with torsion tensor T^{∇}. Then the following conditions are equivalent:*

i). The almost complex structure I^{∇}_1 on the twistor space Z^- of $(M, \mathcal{P})$ is integrable.
ii). The $(0, 2)_J$-part $(T^{\nabla})^{0,2}_J$ of the torsion with respect to all $J \in \mathcal{P}$ vanishes,

$$(T^{\nabla})^{0,2}_J = 0, J \in \mathcal{P}.$$

iii). The almost paracomplex structure P^{∇}_1 on the reflector space Z^+ of $(M, \mathcal{P})$ is integrable.

Proof. We use Proposition 3.5. Since the connection ∇ is a para-quaternionic connection, $\nabla \in \Delta(\mathcal{P})$, the condition (3.18) yields the next expression of the Nijenhuis tensor N_J of any local $J \in \mathcal{P}$,

$$N_J(X,Y) \in span\{J_1X, J_1Y, J_2X, J_2Y, J_3X, J_3Y\}, \tag{3.31}$$

where J_1, J_2, J_3 is an admissible local basis of $\mathcal{P}$.

To prove that ii) implies the integrability of I_1^∇ and P_1^∇, we apply the result of Zamkovoy [36] which states that an almost para-quaternionic 4n-maniofold ($n \geq 2$) is para-quaternionic if and only if the three Nijenhuis tensors N_1, N_2, N_3 satisfy the condition

$$(N_1(X,Y) + N_2(X,Y) - N_3(X,Y)) \in span\{J_1X, J_1Y, J_2X, J_2Y, J_3X, J_3Y\}. \tag{3.32}$$

Clearly, (3.32) follows from (3.31) which shows that the almost para-quaternionic 4n-manifold ($n \geq 2$) $(M,\mathcal{P})$ is a para-quaternionic manifold. Let ∇^0 be a torsion-free para-quaternionic connection on $(M,\mathcal{P})$. Then the almost complex structure $I_1^{\nabla^0}$ on the twistor space Z^- as well as the almost paracomplex structure $P_1^{\nabla^0}$ on the reflector space Z^+ are integrable [26] and $I_1^\nabla = I_1^{\nabla^0}, \quad P_1^\nabla = P_1^{\nabla^0}$ due to Corolarry 3.4.

Hence, the equivalence between i), ii) and iii) is established, which completes the proof. □

From the proof of Proposition 3.5 and Theorem 3.8, we easily derive

Corollary 3.9. *Let ∇ be a para-quaternionic connection on an $4n$-dimensional ($n \geq 2$) almost para-quaternionic manifold $(M,\mathcal{P})$ with torsion tensor T^∇. Then the torsion condition* (3.18) *implies the curvature condition* (3.19).

We note that Corollary 3.9 generalizes the same statement proved in the case of PQKT-connection (see below) in [36] using the first Bianchi identity.

Theorem 3.8 and Corollary 3.4 imply

Corollary 3.10. *Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold.*

- *i). Among the all almost complex structures $I_1^\nabla, \nabla \in \Delta(\mathcal{P})$ on the twistor space Z^- at most one is integrable.*
- *ii). Among the all almost para-complex structures $P_1^\nabla, \nabla \in \Delta(\mathcal{P})$ on the reflector space Z^+ at most one is integrable.*

The proof of the next theorem follows directly from the proof of Theorem 3.8, Theorem 3.7 and Corollary 3.10.

Theorem 3.11. *Let $(M,\mathcal{P})$ be an almost para-quaternionic $4n$-manifold. The next three conditions are equivalent:*

- *1). Either $(M,\mathcal{P})$ is a para-quaternionic manifold (if $n \geq 2$) or $(M,\mathcal{P}=[g])$ is anti-self dual for $n=1$.*
- *2). There exists an integrable almost complex structure I_1^∇ on the twistor space Z^- which does not depend on the para-quaternionic connection ∇.*
- *3). There exists an integrable almost paracomplex structure P_1^∇ on the reflector space Z^+ which does not depend on the para-quaternionic connection ∇.*

4. Para-quaternionic Kähler manifolds with torsion

An almost para-quaternionic Hermitian manifold $(M, \mathcal{P}, g)$ is called para-quaternionic Kähler with torsion (PQKT) if there exists a an almost para-quaternionic Hermitian connection $\nabla^T \in \Delta(\mathcal{P})$ whose torsion tensor T is a 3-form which is (1,2)+(2,1) with respect to each J_a, i.e. the tensor $T(X,Y,Z) := g(T(X,Y),Z)$ is totally skew-symmetric and satisfies the conditions

$$T(X,Y,Z) = -T(J_\alpha X, J_\alpha Y, Z) - T(J_\alpha X, Y, J_\alpha Z) - T(X, J_\alpha Y, J_\alpha Z), \quad \alpha = 1,2;$$
$$T(X,Y,Z) = T(J_3 X, J_3 Y, Z) + T(J_3 X, Y, J_3 Z) + T(X, J_3 Y, J_3 Z).$$

We recall that each PQKT is a quaternionic manifold [36]. The condition on the torsion implies that the (0,2)-part of the torsion of a PQKT connection vanishes. Applying Theorem 3.8, we obtain

Theorem 4.1. *Let $(M, \mathcal{P}, \nabla^T)$ be a PQKT and $\nabla^0 \in \Delta(\mathcal{P})$ be a torsion-free para-quaternionic connection. Then*

i). The almost complex structure $I_1^{\nabla^T}$ on the twistor space Z^- is integrable and therefore it coincides with $I_1^{\nabla^0}$.

ii). The almost paracomplex structure $P_1^{\nabla^T}$ on the reflector space Z^+ is integrable and therefore it coincides with $P_1^{\nabla^0}$.

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